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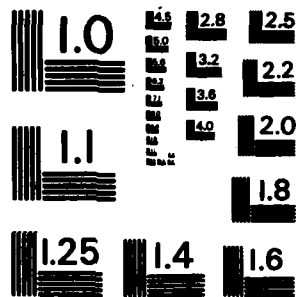
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SMOOTHNESS OF BOUNDED SOLUTIONS OF
NONLINEAR EVOLUTION EQUATIONS

by

Jack K. Hale and Jürgen Scheurle

May 20, 1983

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Finally, generalized Hopf bifurcation in infinite dimensions is considered. Here we discuss smoothness of the bifurcation function and generalize known results on the order of a focus.

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SMOOTHNESS OF BOUNDED SOLUTIONS OF NONLINEAR EVOLUTION EQUATIONS


Jack K. Hale and Jürgen Scheurle

Abstract



It is shown, that in many cases globally defined, bounded solutions of evolution equations are as smooth (in time) as the corresponding operator, even if a general solution of the initial-value problem is much less smooth; i.e., initial values for bounded solutions are selected in such a way that optimal smoothness is attained. In particular, solutions which bifurcate from certain steady states such as periodic orbits, almost-periodic orbits and also homo- and heteroclinic orbits have this property. As examples a neutral functional differential equation, a slightly damped non-linear wave equation, and a heat equation are considered. In the latter case the space variable is included into the discussion of smoothness.

Finally, generalized Hopf bifurcation in infinite dimensions is considered. ^{this document discusses} Here ~~we discuss~~ smoothness of the bifurcation function and generalize known results on the order of a focus.



1. Introduction

Many applications involve initial value problems which can be written as an abstract evolution equation of the form $\frac{du}{dt} = f(u)$ together with an initial condition $u(t_0) = u_0$, where f maps some Banach space E into itself, and $u_0 \in E$. It is well-known that locally there is a unique solution of this problem, if f is defined and Lipschitz continuous in an open neighborhood of u_0 . Furthermore, this solution is as smooth as f . In particular, it is analytic in t , if f is analytic (see Dieudonné [1960]). Unfortunately, in many applications, the domain of f does not contain any open subset; e.g., in case of a partial differential equation, the domain of f could be a subset of smooth functions in some space of continuous functions. In this case, the smoothness of solutions can depend on the initial value in a very sensitive way.

On the other hand, many problems have some kind of invariant sets for the flow induced by the evolution equation, which are attractors for the nearby flow. In this case, one is interested in the solutions on the attractors rather than in the approaching solutions. Thus, the natural question arises if, in general, at least the solutions on the attractors are as smooth as f (cf., Hale, Magalhães and Oliva [1983]).

In this paper, we try to give a partial answer to this question. In several applications, the attractors corresponding to an evolution equation fill a very thin subset in the underlying space E (see e.g., Mañé [1981]), i.e., the initial values for solutions on the attractors are residual. However, if the attractors are bounded, then these solutions are distinguished by being defined and bounded over all of \mathbb{R} . In what follows, we therefore consider

bounded solutions on \mathbb{R} rather than initial value problems. Of course, there are lots of results available which guarantee smoothness of solutions provided the initial data are chosen appropriately, in particular, the well-known Cauchy-Kowalevski Theorem for partial differential equations (see also Ebihara [1978], [1979], and [1982]). But the point is, that we do not a-priori know whether the initial data corresponding to bounded solutions satisfy these conditions. Also, we include smoothness with respect to parameters into our discussion.

Our results are more or less of a local type. The main result gives a positive answer to the question raised above in a neighborhood of certain steady state solutions of nonlinear evolution equations. The linearized equation is supposed to generate a kind of semi-group which is a slight generalization of a continuous semi-group. Roughly speaking, we shall prove that solutions which stay in some neighborhood of the steady state for all t , are as smooth as the equation. This result is of particular interest at bifurcation points, since the neighborhood can be chosen uniformly with respect to small perturbations. Thus bifurcating solutions such as periodic orbits, almost periodic orbits and also homo- and heteroclinic orbits are included. In a sense, this is a supplement to center-manifold theory which is one of the tools used to prove existence for such solutions. Since, in general, center-manifolds are neither analytic nor C^∞ , even if the equation is, the use of center-manifolds does not give an adequate answer to the smoothness question.

In Section 2 we prove some preliminaries. Here we consider equations which have a unique bounded solution on \mathbb{R} (in the non-linear case locally).

We show that this solution is as smooth as the equation, in time as well as with respect to certain parameters. Section 3 contains the main results. Finally, Section 4 is devoted to applications and examples. Our main example is a slightly damped, non-linear wave equation near a critical steady state, where the linearized operator has a simple zero eigenvalue. Here the theory applies to transient wave solutions.

We also consider a non-linear heat equation near a critical steady state. Although, in this case, most solutions are known to be smooth in time, in the analytic case, bounded solutions turn out to have the additional property that they are analytic in both the time and the space variable. For a class of neutral functional differential equations, we prove a global smoothness result for bounded solutions which generalizes a Theorem of Nussbaum [1973].

Finally, we discuss bifurcation of periodic solutions from a focus (Hopf bifurcation) in infinite dimensions. Here a particular problem is smoothness of the bifurcation function. Especially, the unknown frequency which is introduced as a parameter, causes a lot of trouble. Nevertheless, it turns out that the C^k -case ($k \leq \infty$) can be done under a fairly general hypothesis. Here we generalize results of Crandall and Rabinowitz [1977] (cf. also; Neves [1982]). But as far as the order of the focus is concerned, it is of interest to know analyticity of the bifurcation function (see Chow and Hale [1982]). To prove this we need to assume an analyticity condition for the flow generated by the linearized equation. But our conditions are much more general than, e.g., in Kielhöfer [1979].

2. Preliminaries

Let E be a Banach space, and $A : D(A) \subset E \rightarrow E$ a linear, closed operator ($D(A)$ not necessarily dense in E). Assume that the 'exponential' e^{-As} is defined for $s > 0$, i.e., e^{-As} is a family of bounded linear operators on E , which is strongly continuous and has the following property:

$$(i) \quad e^{-A(s_1+s_2)} = e^{-As_1} e^{-As_2}, \quad s_1, s_2 > 0.$$

Also, we assume that there are constants $c > 0$, $b > 0$, and $a \in [0,1)$ such that

$$(ii) \quad \|e^{-As}x\| \leq \frac{ce^{-bs}}{s^a} \|x\|, \quad x \in E.$$

Moreover, for $x \in D(A)$, suppose $e^{-As}x \in D(A)$,

$$(iii) \quad \|e^{-As}x - x\| \rightarrow 0 \text{ as } s \rightarrow 0^+,$$

and the derivative $\frac{d}{ds} e^{-As}x$ exists in the norm-topology of E , where

$$(iv) \quad \frac{d}{ds} e^{-As}x = -Ae^{-As}x = -e^{-As}Ax.$$

Finally, let

$$(v) \quad 0 \in \rho(A),$$

where ρ denotes the resolvent set of A . Because of the decay property (ii), this last condition is not very restrictive. Observe, in the usual sense, e^{-As} is not a continuous semi-group on E , since continuity is only required for $x \in D(A)$.

A very simple example for A is the differential operator $A = -\frac{d}{dx} + b$, $b > 0$, in the space $E = C^0[0, \infty] = \{u: [0, \infty) \rightarrow \mathbb{R} \mid u \text{ is uniformly continuous and bounded}\}$ equipped with the sup-norm, with domain of definition $D(A) = \{u \in E \mid u \text{ is differentiable and } \frac{du}{dx} \in E\}$. Here the exponential is defined by $e^{-As}u(x) = e^{-bs}u(x+s)$ for $u \in E$. In fact, a large number of differential operators together with appropriate boundary conditions and also some functional differential operators have these properties. For more examples, we refer to Section 4 and to the literature: Friedman [1969], Kato [1966], Kielhöfer [1974], Hale [1977], Henry [1981].

Now, consider the equation

$$(2.1) \quad \frac{du}{dt} = -Au + f(t)$$

where f is some function with values in E . Here and in what follows, by solution we mean a C^1 function $u: \mathbb{R} \rightarrow E$ with values in $D(A)$ such that the corresponding equation is satisfied for all t , i.e., a "classical solution". Note that (2.1) has at most one solution which is uniformly bounded for all $t \in \mathbb{R}$. In fact, if there were two different, bounded solutions, then the homogeneous equation $\frac{du}{dt} - Au = 0$ would have a non-trivial bounded solution u . But with $u(t)$ also $u(s+t)$ is a solution of this equation for all $s \in \mathbb{R}$ and, as in the case of a continuous semi-group (see e.g., Kato [1966]), for initial data from $D(A)$ the corresponding Cauchy problem has a unique solution given by the action of e^{-As} . Therefore, we would have $u(s+t) = e^{-As}u(t)$ which, by (ii), and the boundedness of u , implies $u = 0$ which is a contradiction.

Next we solve (2.1) for a class of analytic functions f . More precisely, we assume that f can be continued holomorphically to some complex strip

$$D_\delta = \{t \in \mathbb{C} \mid |\operatorname{Im} t| < \delta, \delta > 0\}.$$

Set

$$C_\delta = \{u: \overline{D_\delta} \rightarrow E \mid u \text{ is continuous, bounded, and holomorphic in } D_\delta\},$$

and introduce the norm

$$|||u||| = \sup_{t \in \overline{D_\delta}} ||u(t)||.$$

Here $\overline{D_\delta}$ denotes the closure of D_δ . Since the limit of a sequence of continuous (holomorphic) functions which converges uniformly on compact subsets of the domain of the functions, is again continuous (holomorphic), C_δ equipped with this norm is a Banach space. Of course, if E is real, then it has to be replaced by its complexification in the definition of C_δ . In this case, it is reasonable to require that $u(t)$ is real if t is, for $u \in C_\delta$. But, for simplicity, we do not change notation. The natural complexifications of the operators A and e^{-At} are also denoted by the same symbols. Note, that properties (i) - (iv) carry over to the complexifications.

Now, define

$$(2.2) \quad \mathcal{J}f(t) = \int_0^\infty e^{-As} f(t-s) ds, \quad f \in C_\delta.$$

2.1 Lemma. The integral operator \mathcal{J} defined by (2.2) is bounded in C_δ . Moreover, for $f \in C_\delta$, $u(t) = \mathcal{J}f(t)$ is the unique solution of equation (2.1).

Proof: Since

$$||\mathcal{J}f|| \leq c \int_0^\infty \frac{e^{-bs}}{s^a} ds ||f||,$$

the integral in (2.2) converges absolutely and uniformly for $t \in \overline{D_\delta}$. Thus, we conclude that $\mathcal{J}f \in C_\delta$ and the operator $\mathcal{J}: C_\delta \rightarrow C_\delta$ is bounded.

It remains to show that $u = \mathcal{J}f$ solves (2.1). We already know that u is continuously differentiable in D_δ (even in the complex sense). So we still have to show $u \in D(A)$ and (2.1) is satisfied. The proof proceeds in three steps.

First we show, for $x \in E$ and $r < t \in \mathbb{R}$, the integral $\int_r^t e^{-A(t-s)} x \, ds$ is contained in $D(A)$, and

$$(2.3) \quad A \int_r^t e^{-A(t-s)} x \, ds = x - e^{-A(t-r)} x$$

holds. Indeed, by (ii) the integral exists, and by (iv) we have

$\frac{d}{ds} e^{-A(t-s)} A^{-1} x = e^{-A(t-s)} x$ for $s < t$. Therefore,

$$\int_r^{t-\varepsilon} e^{-A(t-s)} x \, ds = e^{-A\varepsilon} A^{-1} x - e^{-A(t-r)} A^{-1} x$$

for each $\varepsilon > 0$. Letting $\varepsilon \rightarrow 0$ and using (iii) the claim follows.

Next we prove, for $x \in D(A^2)$

$$(2.4) \quad \frac{1}{h} [e^{-Ah} - I]x \rightarrow -Ax \quad \text{as } h \rightarrow 0^+.$$

Here I denotes the identity operator. Using formula (2.3) with $r = 0$ and $t = h$, and the closedness of A we conclude

$$\frac{1}{h}[e^{-Ah} - I]x = -\frac{1}{h} \int_0^h e^{-A\sigma} Ax d\sigma.$$

Observe, $e^{-A\sigma}Ax$ is a continuous function of σ in $[0, \infty)$. Hence, as $h \rightarrow 0^+$, the right-hand side tends to the value of $-e^{-A\sigma}Ax$ at $\sigma = 0$ which, by (iii), is just $-Ax$. Thus, (2.4) is proved.

Now we are ready to finish the proof. Fix any $t \in D_\delta$. Since $u = \mathcal{L}f$ is continuously differentiable at t , we can use any sequence of difference quotients to approximate $\frac{du}{dt}(t)$. Suppose $h > 0$, and $u_h(t) = \frac{1}{h}[u(t+h) - u(t)]$. We consider $A^{-2}u_h(t)$ rather than $u_h(t)$. By definition and some elementary algebraic manipulations, we have

$$\begin{aligned} A^{-2}u_h(t) &= \frac{1}{h}(e^{-Ah} - I)A^{-2}u(t) \\ (2.5) \quad &+ \frac{1}{h}A^{-2} \int_0^h e^{-A\sigma} f(t) d\sigma \\ &+ \frac{1}{h}A^{-2} \int_0^h e^{-A\sigma} [f(t+h-\sigma) - f(t)] d\sigma. \end{aligned}$$

Here, by (2.4), the first term on the right-hand side tends to $-AA^{-2}u(t)$ as $h \rightarrow 0^+$. Since A^{-2} is bounded on E , the second term can be rewritten as $\frac{1}{h} \int_0^h e^{-A\sigma} A^{-2}f(t) d\sigma$ which, by continuity of the integrand, tends to $A^{-2}f(t)$. The third term, finally, can be estimated by

$$(2.6) \quad c \int_0^h \frac{e^{-b\sigma}}{\sigma^a} d\sigma \cdot \sup_{|\tau-t| \leq h} \left\| \frac{df}{dt}(\tau) \right\|,$$

where c is some positive constant. Here we use the mean-value theorem to estimate $\|f(t+h-\sigma) - f(t)\|$. Evaluating the integral and observing the continuity of the derivative $\frac{df}{dt}$ in D_δ , we see that the expression in (2.6)

tends to 0 as $h \rightarrow 0^+$. But $A^{-2}u_h(t) \rightarrow A^{-2} \frac{du}{dt}(t)$. Hence, from (2.5), we conclude

$$A^{-2} \frac{du}{dt}(t) = -AA^{-2}u(t) + A^{-2}f(t).$$

This implies, $u \in D(A)$ and (2.1) is satisfied. Thus the lemma is proved. \square

2.2 Remarks. (a) Obviously, \mathcal{J} is also bounded considered as an operator in the space $C^0(\mathbb{R}, E)$ or even in $L^\infty(\mathbb{R}, E)$ equipped with the sup-norm. Its norm is the same as in C_0 . For a general $f \in L^\infty(\mathbb{R}, E)$, $u = \mathcal{J}f$ can be considered as a 'mild' solution of (2.1).

If e^{-At} is a continuous semi-group, then the function $u(t)$ is said to be a mild solution of (2.1) if

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-s)}f(s)ds.$$

If $u_0 \in D(A)$ and f is continuously differentiable, it is known that $u(t)$ is a classical solution. Furthermore, if f is sufficiently smooth and u_0 belongs to the domain of some power of A , then u is C^k (see, e.g., Segal [1963]). The argument in the proof of Lemma 2.1 shows that $\mathcal{J}f$ is C^k and is the unique solution of (2.1) bounded for all t , if f and its derivatives up to order k are bounded for all t ($k \geq 1$). Thus, the initial value $u_0 = \int_0^\infty e^{-As}f(s)ds$ for the unique bounded solution is selected in such a way that the solution is as smooth as f .

(b) Lemma 2.1 remains true if, instead of A , the operator $-A$ satisfies (i) - (v), and the integral in the definition of \mathcal{J} is replaced by the integral from 0 to $-\infty$.

As an immediate consequence of the linear theory, we get some results on non-linear problems. Let us now consider the equation

$$(2.7) \quad \frac{du}{dt} = -Au + F(\lambda, u)$$

where A is as above, and F is defined and holomorphic in a (complex) neighborhood U of $(0,0) \in \Lambda \times E$ and takes values in C_δ . Here Λ is another Banach space, and $\lambda \in \Lambda$ is considered to be a parameter. For example, F may be given by $F(\lambda, u) = f(t, \lambda, u)$, where $f: \overline{D}_\delta \times \Lambda \times E \rightarrow E$ is holomorphic. A map between complex Banach spaces is called holomorphic, if it is locally bounded and Gateaux-differentiable (see Hille-Phillips [1959]).

Let us assume that F is uniformly bounded on U . Then, by Remark 2.2(a), as far as solutions in U are concerned, (2.7) is equivalent to the integral equation

$$(2.8) \quad u = \int F(\lambda, u).$$

Furthermore, in analogy to the linear case, a solution u of (2.8) which is contained in $L^\infty(\mathbb{R}, E)$, will be called a 'mild' solution of (2.7).

2.3 Lemma. Let $F: U \subset \Lambda \times E \rightarrow C_\delta$ be holomorphic and uniformly bounded by some constant M , such that $M \cdot ||| \int ||| < r/2$, where r is the radius of some ball $B_r = B_r^\Lambda(0) \times B_r^E(0)$ with $\overline{B}_r \subset U$. Then, for each $\lambda \in \overline{B}_r^\Lambda(0)$, equation (2.7) possesses a solution in $\overline{B}_{r/2}^{C_\delta}(0)$. The solution is unique, even in the class of mild solutions, and depends holomorphically on λ .

Proof: According to the discussion above, it suffices to consider equation (2.8). Let us introduce the map $T: \Lambda \times C_\delta \rightarrow C_\delta$, given by

$(\lambda, u(t)) \mapsto \mathcal{J}[F(\lambda, u(t))(t)]$, whose fixed points we seek. By assumption, for fixed λ , it maps the ball $\bar{B}_{r/2}^{C_\delta}(0)$ into itself. Moreover, it is contracting in this ball. This follows with the aid of Cauchy's integral formula, which shows that $D_u F(\lambda, u)$ is bounded by $2M/r$ for $u \in \bar{B}_{r/2}^E(0)$. Thus, the existence part in the lemma follows by the contraction mapping principle. Furthermore, the map T is holomorphic in $B_r(0,0) \subset \Lambda \times C_\delta$. In fact, again by Cauchy's integral formula, the derivative $DF(\lambda, u)$ is uniformly continuous (the second derivative is bounded) on each ball $\bar{B}_{r-\varepsilon} \subset \Lambda \times E$, $\varepsilon > 0$. This guarantees that T is differentiable in the norm of $\Lambda \times C_\delta$. But now it is a general statement that the fixed points of a holomorphic family of contraction mappings with uniform contraction constant, are holomorphic functions of the parameter (see Chow and Hale [1982, p. 25]).

So it remains to prove uniqueness in the class of mild solutions. To this end we notice that $T(\lambda, \cdot)$ also defines a contraction mapping in the ball $\bar{B}_{r/2}(0)$ of $L^\infty(\mathbb{R}, E)$. Hence, there is at most one fixed point in this ball. Thus the lemma is proved. \square

2.4 Remarks. (a) It is clear from the proof, that Lemma 2.3 is still true for mild solutions, when the space C_δ is replaced by $C^0(\mathbb{R}, E)$ or $L^\infty(\mathbb{R}, E)$.

(b) If the exponential $e^{-\lambda s}$ is holomorphic in some sector $\Gamma = \{s \in \mathbb{C} \mid |\arg s| < \theta, \theta > 0\}$, then the assumptions on F in Lemma 2.3 can be weakened somewhat. Because of the smoothing effect of such exponentials, then it makes sense to replace assumption (ii) by

$$(vi) \quad ||A^\gamma e^{-As} x|| \leq \frac{ce^{-b|s|}}{|s|^{\gamma+a}} ||x|| \quad \text{for all } x \in E,$$

where A^γ is any fractional power of A such that $a + \gamma < 1$ (cf. Kielhöfer [1974]). This implies that the operator \mathcal{J} defined in (2.2) is bounded from C_δ to the Banach space

$$C_\delta^\gamma = \{u \in C_\delta \mid u(t) \in D(A^\gamma), A^\gamma u(t) \in C_\delta \quad \forall t \in \overline{D_\delta}\}$$

equipped with the norm $|||u|||_\gamma = |||u||| + |||A^\gamma u|||$. Here we use the closedness of A^γ . As a consequence, F can have domain in $\Lambda \times C_\delta^\gamma$ and range in C_δ .

(c) Also, if e^{-As} is holomorphic in Γ , then for real s , $e^{-\alpha As}$ is a holomorphic function of α in some complex neighborhood of $\alpha = 1$ and hypotheses (i) - (v) are valid uniformly in α . So, $\mathcal{J}_\alpha: C_\delta \rightarrow C_\delta$ defined by

$$(2.9) \quad \mathcal{J}_\alpha f(t) = \int_0^\infty e^{-\alpha As} f(t-s) ds$$

is a holomorphic function of α and has a uniform bound. Consequently, there is a result similar to Lemma 2.3 for the equation

$$(2.10) \quad \frac{du}{dt} = -\alpha Au + F(\alpha, \lambda, u),$$

where, in particular, the solution is also holomorphic in α . This fact is very important in the theory of Hopf bifurcations, and we will come back to it in Section 4.

Unfortunately, it is not true under our original more general hypotheses. Indeed, for $A = -\frac{d}{dx} + b$ in the above example, $\mathcal{J}_\alpha f(t)$ is defined by

$$\mathcal{J}_\alpha f(t)(x) = \int_0^\infty e^{-\alpha s} f(t-s)(x+\alpha s) ds, \quad x \in [0, \infty).$$

Now, suppose that $f \in C_0$ has the form $f(t)(x) = g(t+x)$, where g is a 2π -periodic function of one complex variable such that

$$\lim_{k \rightarrow \infty} k \sqrt{|g^{(k)}(0)|} = \infty$$

holds for its derivatives. We then claim, that $\mathcal{J}_\alpha f(0)(0)$ is not analytic in α at $\alpha = 1$. A simple example for g is

$$g(z) = \sum_{j=0}^{\infty} e^{-2\delta j} \cos jz.$$

To prove the claim, write

$$\mathcal{J}_\alpha f(0)(0) = \frac{1}{\alpha} h\left(\frac{\alpha-1}{\alpha}\right),$$

where

$$h(\lambda) = \int_0^\infty e^{-b\sigma} g(\lambda\sigma) d\sigma.$$

The formal Taylor series of h at $\lambda = 0$ is easily computed to be

$$h(\lambda) \sim \sum_{k=0}^{\infty} \frac{g^{(k)}(0)}{b^{k+1}} \lambda^k.$$

By assumption, its radius of convergence is zero. Thus the claim follows, since by elementary rules, analyticity of $\mathcal{J}_\alpha f(0)(0)$ at $\alpha = 1$ would imply analyticity of h at $\lambda = 0$.

On the other hand, $\mathcal{J}_\alpha f(t)$ is certainly a C^∞ -function of α in this example. In fact, this is true in the general context - even in the non-

linear case -, when $f(t)$ is of class C^∞ . This is a consequence of the following lemma.

Let $C_u^0(\mathbb{R}, E)$ be the subspace of uniformly continuous functions in $C^0(\mathbb{R}, E)$. It is easily seen that the operator $\mathcal{J}_\alpha: C_u^0(\mathbb{R}, E) \rightarrow C_u^0(\mathbb{R}, E)$ given by (2.9) is point-wise continuous in α and uniformly bounded for α in some neighborhood of 1.

2.5 Lemma. Let $F(\alpha, \lambda, u)(t): B_R^{\mathbb{R}}(1) \times B_R^\Lambda(0) \times B_R^E(0) \times \mathbb{R} \rightarrow E$ be of class C^k ($k \geq 0$), where Λ is finite-dimensional now, and $r < 1$. Suppose that all derivatives are bounded and that the k th derivative is continuous, uniformly with respect to u and t . Furthermore, suppose that the bound M of F satisfies $M < r/2 ||\mathcal{J}_\alpha||$, and that F is Lipschitz continuous with respect to u with uniform constant smaller than $2M/r$. Then, for all $(\alpha, \lambda) \in B_R^{\mathbb{R}}(1) \times B_R^\Lambda(0)$, equation (2.10) has a unique (mild) solution $u \in \overline{B}_{r/2}^{C^0}(0)$, and the function $u(\alpha, \lambda)(t)$ is of class C^k ; the k th derivative of u is continuous uniformly with respect to t .

Proof: As in case of Lemma 2.3, the existence and uniqueness part follows by the contraction mapping principle. Indeed, using the above properties of \mathcal{J}_α and the assumptions on F , we find that the map $T^0: B_R^{\mathbb{R}}(1) \times B_R^\Lambda(0) \times \overline{B}_{r/2}^{C_u^0} \rightarrow \overline{B}_{r/2}^{C_u^0}(0)$ defined by $(\alpha, \lambda, u) \mapsto \mathcal{J}_\alpha[F(\alpha, \lambda, u)]$ is continuous and a uniform contraction with respect to $u(t)$. Thus, for each α and λ , the unique fixed point $u(\alpha, \lambda)$ of T^0 is a mild solution of (2.10). It is a uniformly continuous function of t and depends continuously on α and λ . When $k = 0$, this proves the lemma.

Now, suppose $k = 1$. Then the derivatives $D_\lambda u(\alpha, \lambda)$, $D_t u(\alpha, \lambda)$, and $D_\alpha u(\alpha, \lambda)$ can be determined as fixed points of operators

$$T^1: B_T^R(1) \times B_T^A(0) \times \mathcal{D}(\Lambda, C_u^0(\mathbb{R}, E)) \rightarrow \mathcal{D}(\Lambda, C_u^0(\mathbb{R}, E)),$$

$$T^2: B_T^R(1) \times B_T^A(0) \times \mathcal{D}(\mathbb{R}, C_u^0(\mathbb{R}, E)) \rightarrow \mathcal{D}(\mathbb{R}, C_u^0(\mathbb{R}, E)),$$

$$T^3: B_T^R(1) \times B_T^A(0) \times \mathcal{D}(\mathbb{R}, C_u^0(\mathbb{R}, E)) \rightarrow \mathcal{D}(\mathbb{R}, C_u^0(\mathbb{R}, E)),$$

where $\mathcal{D}(V, W)$ denotes the space of linear bounded operators between Banach spaces V and W equipped with the uniform norm topology. Taking derivatives of the identity $u(\alpha, \lambda)(t) = T^0(\alpha, \lambda, u(\alpha, \lambda))(t)$, we find the T^k ($k = 1, 2, 3$) to be given by

$$T^1(\alpha, \lambda, \eta)\tilde{\lambda} = \mathcal{J}'_{\alpha}[D_{\lambda}F(\alpha, \lambda, u(\alpha, \lambda))\tilde{\lambda} + D_uF(\alpha, \lambda, u(\alpha, \lambda))N\tilde{\lambda}], \quad \tilde{\lambda} \in \Lambda,$$

$$T^2(\alpha, \lambda, M)\tilde{t} = \mathcal{J}'_{\alpha}[D_tF(\alpha, \lambda, u(\alpha, \lambda))(t)\tilde{t} + D_uF(\alpha, \lambda, u(\alpha, \lambda))M\tilde{t}], \quad \tilde{t} \in \mathbb{R},$$

and

$$T^3(\alpha, \lambda, K)\tilde{\alpha} = \mathcal{J}'_{\alpha}\left[-\frac{1}{\alpha}F(\alpha, \lambda, u(\alpha, \lambda))\tilde{\alpha} + D_{\alpha}F(\alpha, \lambda, u(\alpha, \lambda))\tilde{\alpha} + D_uF(\alpha, \lambda, u(\alpha, \lambda))K\tilde{\alpha}\right] + \tilde{\alpha} \in \mathbb{R}.$$

$$\mathcal{J}'_{\alpha}[D_uF(\alpha, \lambda, u(\alpha, \lambda))D_tu(\alpha, \lambda)(t)\tilde{\alpha} + D_tF(\alpha, \lambda, u(\alpha, \lambda))(t)\tilde{\alpha}],$$

where $\mathcal{J}'_{\alpha}: C_u^0(\mathbb{R}, E) \rightarrow C_u^0(\mathbb{R}, E)$ is given by

$$\mathcal{J}'_{\alpha}f(t) = \frac{1}{\alpha} \int_0^{\infty} e^{-\alpha s} f(t-s) ds.$$

For T^3 , we have used the representation

$$\mathcal{J}_{\alpha}f(t) = \frac{1}{\alpha} \int_0^{\infty} e^{-As} f(t - \frac{1}{\alpha}s) ds,$$

since e^{-As} is not supposed to be differentiable in s . Note, that \mathcal{J}'_{α}

has similar properties as \mathcal{J}_α . Thus, we conclude that the mappings T^k are continuous. Here we use the assumptions on F and, since \mathcal{J}_α and \mathcal{J}'_α are only point-wise continuous in α , the fact that $\tilde{\lambda}, \tilde{t}$, and $\tilde{\alpha}$ belong to finite-dimensional spaces. In case of T^3 , it is assumed that $D_t u(\alpha, \lambda)(t) \in \mathcal{D}(\mathbb{R}, C_u^0(\mathbb{R}, E))$ depends continuously on α and λ which has to be shown first. Furthermore, the uniform Lipschitz continuity of F implies $\|D_u F(\alpha, \lambda, u)\| < 2M/r$. Therefore, each T^k is a uniform contraction with respect to its third argument.

Thus, as in case of T^0 , we conclude the existence of unique fixed points N, M and K , respectively, which depend continuously on α and λ . It is not very hard to show that these fixed points are actually the derivatives $D_\lambda u(\alpha, \lambda)$, $D_t u(\alpha, \lambda)(t)$, and $D_\alpha u(\alpha, \lambda)$ (cf. Chow and Hale [1982, pp. 25]). But this proves the lemma for $k = 1$.

Now it should be clear how to proceed for any $k < \infty$ by induction. Since this induction is straight-forward, it is left to the reader. \square

2.6 Remark. In λ and t , the function u in Lemma 2.5 is that smooth even when the dimension of Λ is infinite. In λ alone, it is C^k even when the smoothness assumption on the k th derivative of F is weakened to continuity plus uniform continuity with respect to u and t in λ and u only. As far as smoothness of u in α, λ , and t is concerned, the lemma is still true for infinite dimensional Λ , provided one considers directional derivatives with respect to λ rather than Frechét derivatives.

3. Main Results

In this section, we consider a system of autonomous equations

$$(3.1) \quad \begin{aligned} \frac{dx}{dt} &= Bx + u(\lambda, x, y) \\ \frac{dy}{dt} &= -Ay + v(\lambda, x, y). \end{aligned}$$

Here the variables x and y run in (complex) Banach spaces F and E , respectively, and, as before, $\lambda \in \Lambda$ is considered as a parameter. We assume that A has the same properties as in Section 2, and that B is a bounded, linear operator in the space F . The functions u and v are supposed to be bounded and holomorphic from some neighborhood V of $(0,0,0) \in \Lambda \times F \times E$ to F and E , respectively. (If u and v are given as real analytic maps, then we consider holomorphic extensions (cf. Hille and Phillips [1969])). Moreover, the bound of v is supposed to be less than $r/2 ||| \mathcal{J} |||$ in some ball $\bar{B}_r = \bar{B}_r^\Lambda(0) \times \bar{B}_r^F(0) \times \bar{B}_r^E(0) \subset V$.

We shall show that each solution of (3.1) which is defined for real t and contained in the ball $B_{r/2}$ has a holomorphic continuation into some complex domain D_δ . First we note some simple lemmas.

3.1 Lemma. The map $\tilde{v}: \Lambda \times C^0(\mathbb{R}, F) \times E \rightarrow C^0(\mathbb{R}, E)$ given by $\tilde{v}(\lambda, x, y)(t) = v(\lambda, x(t), y)$ is holomorphic in the ball $\tilde{B}_r = B_r^\Lambda(0) \times B_r^{C^0}(0) \times B_r^E(0)$. Moreover, it is strictly bounded by $r/2 ||| \mathcal{J} |||$.

Proof: By Cauchy's integral formula, Dv is bounded and uniformly continuous (D^2v is bounded) on each ball $\bar{B}_{r-\varepsilon}$, $\varepsilon > 0$. This implies differentiability of \tilde{v} in the norm-topology of $C^0(\mathbb{R}, E)$. The bound on \tilde{v} follows from the bound on v . \square

From this lemma and Remark 2.4(a), we conclude that for each $\lambda \in B_r^\Lambda(0)$ and $x \in B_r^{C^0}(0)$, the equation $y = \mathcal{F}\tilde{V}(\lambda, x, y)$ has a unique solution $y(t) = y^*(\lambda, x)(t)$ in $B_{r/2}^E(0)$. That is, we consider λ and x as parameters. The map $y^*: B_r^\Lambda(0) \times B_r^{C^0}(0) \rightarrow C^0(\mathbb{R}, E)$ is holomorphic.

3.2 Lemma. The map $\tilde{u}: \Lambda \times C^0(\mathbb{R}, F) \rightarrow C^0(\mathbb{R}, F)$ given by $\tilde{u}(\lambda, x)(t) = u(\lambda, x(t), y^*(\lambda, x)(t))$ is holomorphic in the ball $\tilde{B}_r = B_r^\Lambda(0) \times B_r^{C^0}(0)$.

Proof: The proof is analogous to the one of the preceding lemma. \square

Now we come to the crucial point for the later argument. Namely, we want to consider the initial value problem

$$(3.2) \quad \begin{aligned} \frac{dx}{ds} &= Bx + \tilde{u}(\lambda, x) \\ x(0) &= x_0 \end{aligned}$$

in the space $C^0(\mathbb{R}, F)$. This kind of trick has already been used by Nussbaum [1973] in a similar context. Since, by assumption, $(Bx)(t) = Bx(t)$ defines a bounded, linear operator on this space, and by Lemma 3.2, the right-hand side in the differential equation is holomorphic in \tilde{B}_r . Therefore, we know (cf. Dieudonné [1960, Theorem 10.4.5]), there exists a $\delta > 0$ and a holomorphic mapping

$$x^*: B_r^\Lambda(0) \times B_{r/2}^{C^0}(0) \times B_\delta^E(0) \rightarrow B_r^{C^0}(0)$$

such that $x(s) = x^*(\lambda, x_0, s)$ is the unique solution of (3.2) for $|s| < \delta$. Note, s runs in a complex neighborhood of $s = 0$, but for each s , $x(s)$ is a function of a real variable which we call t .

Now we are ready to prove

3.3 Theorem. Let A be as in Section 2, and let $B: F \rightarrow F$ be a bounded linear operator. The mappings $u: V \rightarrow F$ and $v: V \rightarrow E$ are assumed to be holomorphic and bounded. The bound of v is required to be less than $r/2 ||| \mathcal{J} |||$. Now, let $(\lambda, x_0(t), y_0(t))$ be a (classical) solution of (3.1) with $\lambda \in B_r^A(0)$, $x_0(t) \in B_{r/2}^F(0)$, and $y_0(t) \in B_{r/2}^E(0)$ for all $t \in \mathbb{R}$. Then the functions x_0 and y_0 have holomorphic extensions defined on some complex strip D_δ ; δ depends on the solution.

Proof: By assumption, $x_0 \in B_{r/2}^{C^0}(0)$. Therefore, we can solve problem (3.2) with x_0 as initial value. According to the discussion above, there is a unique solution $x^*(\lambda, x_0, s) \in B_r^{C^0}(0)$ for s in some complex ball $B_\delta^C(0)$. We claim

$$x^*(\lambda, x_0, s)(t) = x_0(s+t),$$

when s is real. To prove this claim, we show that ξ given by $\xi(s)(t) = x_0(s+t)$, is a solution of (3.2), and we use uniqueness. We clearly have $\xi(s) \in C^0(\mathbb{R}, F)$ and $\xi(0) = x_0$.

From the first equation in (3.1), we see that, with x_0 , also its derivative is bounded on \mathbb{R} . Hence, x_0 is certainly uniformly continuous. Furthermore, as a classical solution of the corresponding differential equation, $y_0 \in B_{r/2}^{C^0}(0)$ is the unique solution of the integral equation $y = \mathcal{F}\tilde{v}(\lambda, x_0, y)$, where \tilde{v} has been defined in Lemma 3.1; i.e., we have $y_0 = y^*(\lambda, x_0)$. Note, if x_0 is uniformly continuous, so is $\tilde{v}(\lambda, x_0, y)$ for all λ and y . Thus, y_0 is also uniformly continuous (cf. Lemma 2.5).

Now, with x_0 and y_0 , also the function $Bx_0(t) + u(\lambda, x_0(t), y_0(t))$ is uniformly continuous. So we can again use the first equation in (3.1)

to conclude: $\frac{dx_0}{dt}$ is not only bounded but also uniformly continuous on \mathbb{R} .

This, in turn, implies that the function $\xi: \mathbb{R} \rightarrow C^0(\mathbb{R}, F)$ is continuously differentiable in the C^0 -norm. Its derivative at s is given by

$$\frac{d\xi}{ds}(s)(t) = \frac{dx_0}{dt}(s+t).$$

So, in order to prove that the differential equation in (3.2) is satisfied, we still have to show $\tilde{u}(\lambda, \xi(s))(t) = u(\lambda, x_0(s+t), y_0(s+t))$.

This amounts to proving

$$y^*(\lambda, \xi(s))(t) = y_0(s+t).$$

But this can be easily verified for each s , as we already did for $s = 0$ before. Thus, the claim follows.

Now we are almost done. In fact, since $x^*(\lambda, x_0, \cdot): B_\delta^{\mathbb{C}}(0) \rightarrow B_r^{C^0}(0)$ is holomorphic and $x_0(s+t) = x^*(\lambda, x_0, s)(t)$ for real s , $x^*(\lambda, x_0, s)(t)$ defines a holomorphic extension of x_0 in each ball $B_\delta^{\mathbb{C}}(t)$, $t \in \mathbb{R}$. In every intersection of two of these balls, which contains a whole interval of the real line, the corresponding extensions agree by standard results on holomorphic continuation. Thus, we have actually defined a holomorphic extension of x_0 on the whole strip D_δ . In particular, it is bounded and therefore contained in some space C_δ .

This, in turn, implies that $\tilde{v}(\lambda, x_0, y)$ is a holomorphic function of λ and y with values in C_δ . Thus, it follows from Lemma 2.3 that also y_0 has an extension contained in C_δ . But this proves the theorem. \square

3.4 Remark. According to Remark 2.2(b), Theorem 3.3 remains true, if in the second equation in (3.1) the operator A is replaced by $-A$. Also, we can add a third equation of this type.

Finally, we state an analogous result for the C^k -case.

3.5 Theorem. Let A and B be as in Theorem 3.3. Suppose that the mappings u and v are of class C^k ($k \geq 1$) and bounded in their domain together with all derivatives up through order k. The kth derivatives are assumed to be uniformly continuous in x and y. Furthermore, suppose that the bound M of v satisfies $M < r/2||\mathcal{J}||$ and that v is Lipschitz continuous with respect to y with uniform constant smaller than $2M/r$, where r is the radius of some ball strictly contained in the domain of u and v. Then the x- and the y-component of a (classical) solution of (3.1) with properties as stated in Theorem 3.3, are actually C^k functions of t. (The kth derivatives are uniformly continuous.)

Proof: The proof is completely analogous to the one of Theorem 3.3. Hence, we only give an outline. First one solves the integral equation $y = \mathcal{J}\tilde{v}(\lambda, x_0, y)$ for y as a function of x_0 . By Remark 2.6, this function is C^k from $C^0(\mathbb{R}, F)$ into $C^0(\mathbb{R}, E)$. Then one considers the associated initial value problem (3.2) in $C^0(\mathbb{R}, F)$, and shows that $\xi(s)$ given by $\xi(s)(t) = x_0(s+t)$ is a solution. Since the right-hand side of the differential equation is of class C^k ($k \geq 1$), one knows that the solution is unique and contained in $C^0(\mathbb{R}, C^0(\mathbb{R}, F))$. But this implies that x_0 is C^k and its kth derivative is uniformly continuous. Now one can again use the integral equation $y = \mathcal{J}\tilde{v}(\lambda, x_0, y)$ to conclude by Lemma 2.5, that y_0 is just as smooth. \square

Roughly speaking, the Theorems 3.3 and 3.5 can be rephrased as follows:

"Small" bounded solutions of (3.1) are as smooth as the mappings u and v , which is again a manifestation of some kind of relationship between boundedness on \mathbb{R} and smoothness of solutions of evolution equations.

4. Applications and examples

A typical situation where a system of equations of type (3.1) occurs, are bifurcation problems. One considers a (non-linear) evolution equation which has a known steady-state solution for all values of some scalar parameter λ . This steady-state solution is supposed to be stable in the linearized sense for all values of λ less than some critical one, say $\lambda = 0$. At $\lambda = 0$, some modes of the linearized equation are going to be unstable, i.e., there are some purely imaginary eigenvalues of the corresponding operator. If the remaining spectral part is strictly bounded away from the imaginary axis, then there is a corresponding decomposition of this operator and the underlying space, given by so-called Dunford projections (for details see Kato [1969]). In many cases, this leads to an equivalent system of equations which, locally, fits into our theory. The space F corresponds to the pure imaginary spectral part and can often be identified with a finite-dimensional space.

In this latter case, of course, one has a finite-dimensional center-manifold which depends continuously on the parameter (cf. Marsden and McCracken [1976], Chow and Hale [1982]). It is known that the bounded solutions in a neighborhood of the critical steady-state solution lie on any center-manifold. Thus, restricting the original equations to a center-manifold, the problem is reduced to a finite-dimensional evolution equa-

tion, where smoothness is clear. However, smoothness of center-manifolds is quite a problem. For example, for analytic equations, there need not be any analytic center-manifold in general (see Carr [1981]). Nevertheless, Theorem 3.3 applies and guarantees analyticity of the bounded solutions. These include all relevant bifurcating solutions such as periodic orbits, almost periodic solutions and also homoclinic and heteroclinic trajectories. Furthermore, for C^∞ -equations, Theorem 3.5 implies that in a neighborhood of the critical steady state, these solutions are C^∞ functions of time. This is also not obvious from center-manifold theory.

To illustrate these observations, we consider a slightly damped nonlinear wave equation together with zero boundary conditions:

$$(4.1) \quad \begin{aligned} \partial_{tt} u - \partial_{\xi\xi} u + \delta \partial_t u - u - \lambda u + f(u) &= 0, & \xi \in (0, \pi) \\ u(0, t) = u(\pi, t) &= 0. \end{aligned}$$

Here δ is some small, positive constant, λ a real parameter varying in a neighborhood of 0, and $f: \mathbb{R} \rightarrow \mathbb{R}$ is supposed to be analytic and of order $o(|u|)$ as $u \rightarrow 0$. Let $X = \dot{H}^1[0, \pi] \otimes L^2[0, \pi]$ be the product of the usual Sobolev spaces. Elements of X are denoted by $w = (u, v)$. Introduce operators

$$\tilde{A} = \begin{pmatrix} 0 & 1 \\ \frac{d^2}{d\xi^2} + 1 & -\delta \end{pmatrix}, \quad D(\tilde{A}) = (H^2[0, \pi] \cap \dot{H}^1[0, \pi]) \otimes \dot{H}^1[0, \pi]$$

and

$$G(\lambda, w) = \begin{pmatrix} 0 \\ \lambda u - f(u) \end{pmatrix}$$

in X . Obviously, \tilde{A} is a densely defined, closed, linear operator and G is analytic. Here we use Sobolev's embedding theorem $H^1[0, \pi] \subset C^0[0, \pi]$. Thus, (4.1) can be rewritten as the evolution equation

$$(4.2) \quad \frac{d}{dt} w = -\tilde{A}w + G(\lambda, w), \quad w \in X.$$

Representing w as a Fourier series $w(x) = \sum_{k=1}^{\infty} w_n \sin kx$, we find that the spectrum of $-\tilde{A}$ consists of the eigenvalues

$$\mu_{1/2}^k = -\frac{\delta}{2} \pm \sqrt{\frac{\delta^2}{4} + 1 - k^2}, \quad k = 1, 2, 3, \dots$$

Thus, there is one eigenvalue zero, and all the others have negative real part. All eigenvalues are simple; in particular, the eigenspace of the zero eigenvalue is spanned by $\begin{pmatrix} 0 \\ \sin \xi \end{pmatrix}$. Call this eigenspace F and the complementary one E . Let P and Q be the corresponding eigenprojections. Set $Pw = x$ and $Qw = y$. Then (4.2) can be rewritten in the form (3.1), where $B = 0$, $A = \tilde{A}|_E$, $u = PG$, and $v = QG$. By assumption, u and v are of order $o(|\lambda| + ||x|| + ||y||)$ as $\lambda \rightarrow 0$, $x \rightarrow 0$, and $y \rightarrow 0$. Furthermore, again by Fourier series, it can be shown that the interval $(-\delta/2, \infty)$ belongs to the resolvent set of $-A$, and

$$||(A+\mu)^{-1}|| \leq \frac{1}{\mu + \delta/2}, \quad \mu > -\delta/2.$$

Therefore, by a well-known theorem on semi-groups (see e.g., Kato [1966]), $-A$ generates a continuous semi-group in E , and (ii) is satisfied with $a = 0$, $b = \delta/2$, and $c = 1$. Thus, all hypotheses of Theorem 3.3 are fulfilled for sufficiently small r .

Using center-manifold theory, it can be shown that there is a saddle-node bifurcation of steady states of (4.1) at $\lambda = 0$. If, for example, the second-order term in the Taylor expansion of f is non-degenerate, then, for each λ near zero, there is exactly one non-trivial steady-state solution which is connected with the trivial one by a heteroclinic orbit, i.e., a transient wave. Here one uses the fact that center-manifolds are one-dimensional in this example, invariant under the corresponding flow and contain all steady states. By Theorem 3.5, the transient wave solutions are analytic in time.

Qualitatively, the same result holds true for the parabolic problem

$$(4.3) \quad \begin{aligned} \partial_t u - \partial_{\xi\xi} u - u - \lambda u + f(u) &= 0, \quad \xi \in (0, \pi) \\ u(0, t) = u(\pi, t) &= 0, \end{aligned}$$

where λ and f are as before. This problem can be written as an evolution equation in $X = C^0[0, \pi]$. The linear operator

$$\tilde{A} = \frac{d^2}{d\xi^2}, \quad D(A) = \{u \in C^2[0, \pi] / u(0) = u(\pi) = 0\}$$

has simple eigenvalues $\mu^k = 1 - k^2$, $k = 1, 2, 3, \dots$. Corresponding eigenfunctions are $\sin kx$. Using Green's function, we see that the restriction of \tilde{A} to the eigenspace E corresponding to the nonzero eigenvalues, which we denote by $-A$, satisfies the Hille-Yoshida condition for a generator of a holomorphic semi-group, except that $D(-A)$ is not dense in E (for the Hille-Yoshida condition see, e.g., Hille and Phillips [1959]). But nevertheless,

the exponential e^{-As} can be defined by Dunford's integral, and hypotheses (i) - (v) are satisfied. In addition, e^{-As} is holomorphic in the sense of Remark 2.4(b).

Furthermore, the same result is true in a space of holomorphic functions

$$Y = \{u: \bar{K} \rightarrow \mathbb{C} \mid u \text{ is continuous and holomorphic in } K\}$$

where

$$K = \{\xi \in \mathbb{C} \mid |\arg \xi|, |\arg(\pi - \xi)| < \epsilon, \xi \neq 0, \pi\}$$

for some $\epsilon \in (0, \pi/4)$ (see Scheurle [1982]). This has an interesting consequence. Since the eigenspace F to the zero eigenvalue of $-\tilde{A}$ is spanned by the holomorphic function $\sin \xi$, it follows that any classical solution of (4.3) which remains in some neighborhood of $\lambda = 0, u = 0$ for all t , is not only analytic in time, but also in the space-variable. In fact, if (4.3) is rewritten as a system (3.1), then, for given x with values in Y the unique solution of the second equation takes values in Y too. This implies, in particular, that bifurcating steady states are analytic functions of ξ .

Also, note that in case of the parabolic equation, there are only a finite number of eigenvalues of $-\tilde{A}$ to the right of each line parallel to the imaginary axis in the complex plane. Thus, we get an arbitrarily large decay rate b for e^{-As} , if $-A$ is the spectral part of $-\tilde{A}$ belonging to the eigenvalues outside a sufficiently large ball. This, in turn, leads to an arbitrarily small norm for the corresponding operator \mathcal{L} .

On the other hand, the complementary spectral part of \tilde{A} is still a bounded operator on a finite dimensional space. Rewriting (4.3) in form (3.1) according to such spectral decompositions, it should be possible to get some kind of global smoothness result for bounded solutions out of the present theory, since the smallness condition required for the mapping v in (3.1) is proportional to $1/||\mathcal{L}||$. But we will not work out the details here.

Rather, we want to consider a class of functional differential equations globally. In fact, Nussbaum [1973] used the present ideas first to prove analyticity of bounded solutions of the equation

$$(4.4) \quad \frac{dy}{dt}(t) = f(y_t)$$

in \mathbb{R}^n , where y_t is defined by $y_t(s) = y(t+s)$ for all s .

Let us briefly indicate a proof of analyticity for the neutral case

$$(4.5) \quad \frac{d}{dt} Dy_t = f(y_t),$$

where $D: C = C^0([-r, 0], \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is linear, bounded, atomic at zero, and the solutions of $Dy_t = 0$ satisfy

$$|y_t| \leq ce^{-bt}|\phi|, \quad t \geq 0,$$

where $y_0 = \phi$ and $b, c > 0$ (i.e., D is stable). See Hale (1977) for the basic existence theory. Let X be the complex Banach space of functions $x: (-\infty, +\infty) \rightarrow \mathbb{C}^n$, which are uniformly continuous, equipped with the sup-norm, and let $U = \{x \in X / \sup_t |\operatorname{Im} x(t)| < h\}$ be a neighborhood of the real subspace of X . Suppose that f has a holomorphic extension to U

which takes closed bounded subsets of U into bounded subsets of \mathbb{C}^n .

Also, extend D to X .

Now, define $\tilde{D}: X \rightarrow X$ and $F: U \rightarrow X$ by $(\tilde{D}(x))(t) = D(x_t)$ and $(F(x))(t) = f(x_t)$, respectively, for all $t \in \mathbb{R}$. The linear operator \tilde{D} is actually an isomorphism in X . Its inverse has a representation

$$(\tilde{D}^{-1}x)(t) = \int_{-\infty}^0 d\eta(\theta)x(t+\theta),$$

where η has bounded variation (see Hale [1977]). F is holomorphic.

Thus, $F \circ \tilde{D}^{-1}$ is holomorphic in some neighborhood U .

Now, consider the ordinary differential equation

$$(4.6) \quad \frac{d}{ds} x = F(\tilde{D}^{-1}x)$$

in U . If y^0 is a (classical) solution of (4.5) bounded on \mathbb{R} , then define $\xi: \mathbb{R} \rightarrow X$ by $\xi(s) = \tilde{D}y_s^0$, i.e., $\xi(s)(t) = D(y_{s+t}^0)$. Using (4.5), we find by boots trapping, that Dy_t^0 and its derivative are contained in X as functions of t . Therefore, ξ is well-defined and continuously differentiable, $\xi(0) = \tilde{D}y^0$. By definition, we have

$$\frac{d\xi}{ds}(s)(t) = \frac{d}{ds} D(y_{s+t}^0) = f(y_{s+t}^0) = F(y_s^0)(t) = F(\tilde{D}^{-1}\xi(s))(t)$$

for all $s, t \in \mathbb{R}$. Hence, ξ is a solution of (4.6) with initial value $\tilde{D}y^0$. Now we can argue as in the proof of Theorem 3.3 to conclude that $x_0 = \tilde{D}y^0$ has a holomorphic extension contained in some space C_δ . It then follows that y^0 is analytic.

If f is of class C_u^k on each bounded subset of the real subspace of X , then it follows in the same way, that globally defined, bounded solutions of (4.5) are C_u^k functions.

Finally, we talk about bifurcation of periodic solutions from a focus (Hopf bifurcation), which was actually the starting point for the whole paper. A focus is a steady-state solution for which the linearized operator has a pair of simple, purely imaginary eigenvalues, say $\pm i\omega_0$, and all the other spectral points are off the imaginary axis. One looks for periodic solutions in a small neighborhood of the focus. For a finite-dimensional vector field which is analytic, it can be shown that there is either a 'vertical' bifurcation of periodic orbits at the critical parameter value or there are at most a finite number of branches of periodic orbits emanating at the critical parameter value. The maximal number of possible branches is said to be the order of the focus (see Chow and Hale [1982]). The crucial point is to prove analyticity of the bifurcation function which is a function of λ , a and α , where λ is the parameter, a measures the amplitude of the bifurcating orbits, and α is the reciprocal of the unknown frequency ω .

In what follows, we describe a generalization to infinite dimensions. Suppose that the equation is written in form (3.1), where the focus is at $\lambda = 0$, $x = 0$, $y = 0$; F is the eigenspace corresponding to the eigenvalues $\pm i\omega_0$ and can therefore be identified with \mathbb{R}^2 ; E is the complementary eigenspace, and B and $-A$ are the corresponding spectral parts of the linearized operator. The operator $-A$ is supposed to satisfy hypotheses (i) - (v). In x -space, we can introduce coordinates (x_1, x_2) such that B is given by the matrix

$$B = \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}.$$

Furthermore, it is convenient to introduce polar coordinates

$$x_1 = p \cos \theta, \quad x_2 = -p \sin \theta, \quad y = pw.$$

This transformation can be justified using center-manifold theory (see Chow and Hale [1982]). It leads to the following set of equations

$$\begin{aligned} \frac{d\theta}{dt} &= \omega_0 - (u_1 \sin \theta + u_2 \cos \theta)/p \\ (4.7) \quad \frac{dp}{dt} &= u_1 \cos \theta - u_2 \sin \theta \\ \frac{dw}{dt} &= -Aw + v/p - (u_1 \cos \theta - u_2 \sin \theta)w/p, \end{aligned}$$

where all functions are evaluated at $(\lambda, p \cos \theta, -p \sin \theta, pw)$, and $u = (u_1, u_2)$. Now we rescale time by $\omega t \mapsto t$, set $\alpha = \omega^{-1}$, and look for 2π -periodic solutions of the form

$$\begin{aligned} \theta &= t + \theta^*(\lambda, a, \alpha, t), \quad p = a + a \cdot p^*(\lambda, a, \alpha, t), \\ w &= w^*(\lambda, a, \alpha, t), \end{aligned}$$

where we require $\int_0^{2\pi} p^*(\lambda, a, \alpha, t) dt = 0$. The resulting equations for θ^* , p^* , and w^* are written as

$$\begin{aligned} \frac{d\theta^*}{dt} &= \Theta(\lambda, a, \alpha, t, \theta^*, p^*, w^*) - G_1(\lambda, a, \alpha) \\ (4.8) \quad \frac{dp^*}{dt} &= R(\lambda, a, \alpha, t, \theta^*, p^*, w^*) - G_2(\lambda, a, \alpha), \\ \frac{dw^*}{dt} &= -\alpha Aw + W(\lambda, a, \alpha, t, \theta^*, p^*, w^*) \end{aligned}$$

where

$$(4.9) \quad G_1 = \frac{1}{2\pi} \int_0^{2\pi} \Theta \, dt, \quad G_2 = \frac{1}{2\pi} \int_0^{2\pi} R \, dt.$$

Thus, if u and v are of class C_u^k (analytic) and if $u(\lambda, 0, 0) = 0$, $v(\lambda, 0, 0) = 0$, then Θ , R and W are 2π -periodic in t , of class C_u^{k-1} (analytic), and of order $O(|\lambda| + |a| + |\frac{1}{\omega_0} - \alpha| + |p^*|^2 + |w^*|^2)$ as $\lambda \rightarrow 0$, $a \rightarrow 0$, $\alpha \rightarrow \omega_0^{-1}$, $p \rightarrow 0$, and $w \rightarrow 0$, uniformly in t and θ^* . The function $G = (G_1, G_2)$ is the bifurcation function.

Now we associate a fixed point problem to (4.8). To this end, we introduce the operator \tilde{J} , which maps a continuous, 2π -periodic function $f(t)$ with mean-value zero to the unique solution of the equation $\frac{du}{dt} = f(t)$ with mean-value zero. Also, let Q denote the projection, which maps $f(t)$ to its mean-value, $P = I - Q$. Thus, (4.8) can be rewritten as

$$(4.10) \quad \begin{aligned} \theta^* &= \tilde{J}P\Theta(\lambda, a, \alpha, t, \theta^*, p^*, w^*) \\ p^* &= \tilde{J}PR(\lambda, a, \alpha, t, \theta^*, p^*, w^*), \\ w^* &= \tilde{J}_\alpha W(\lambda, a, \alpha, t, \theta^*, p^*, w^*) \end{aligned}$$

with \tilde{J}_α as in (2.9).

Note that P is continuous in the space $C_{2\pi}^0(\mathbb{R}, \mathbb{R})$ and \tilde{J} is bounded on the range of P . Also, both P and \tilde{J} commute with differentiation on subsets of continuously differentiable functions. Therefore, in the C^k -case, we can apply the technique of the proof of Lemma 2.5 to (4.10) to conclude that for (λ, a, α) near $(0, 0, \omega_0^{-1})$ there is a

unique solution θ^*, p^*, w^* which is a C^{k-1} function of λ, a, α , and t . This then implies that the bifurcation function G is C^{k-1} . Note, that this has to be understood in the sense of directional derivatives with respect to λ if Λ has infinite dimension.

Furthermore, if we assume that e^{-As} is holomorphic, then we can use Remark 2.4(c) and the method of Lemma 2.3 to prove analyticity of the bifurcation function in the analytic case. Thus we have the same alternative of possible bifurcations from the focus as in the finite-dimensional case. But, as already pointed out in Remark 2.4, for more general equations the problem is still open.

To summarize, we have outlined the proof of the following theorem:

4.1 Theorem. Consider the equation

$$(4.11) \quad \frac{du}{dt} = Cu + f(\lambda, u),$$

where $u \in X$, a Banach space, and $\lambda \in \Lambda$, another Banach space. Suppose that $f: \Lambda \times X \rightarrow X$ is of class C_u^k ($k \geq 1$) (analytic) satisfying $f(\lambda, 0) = 0$ for all λ and $D_u f(0, 0) = 0$ and that C is a closed linear operator in X with a pair of simple eigenvalues $\pm i\omega_0$ and the other spectrum contained in the left complex half plane. Let the negative part of C , say $-A$, satisfy the hypotheses (i) - (v) of Section 2. Then, the problem of finding periodic orbits of (4.11) near $\lambda = 0, u = 0$ is equivalent to solving the bifurcation equation $G = 0$. The function G with components G_1 and G_2 given in (4.9) is of class C^{k-1} (analytic, if e^{-As} is holomorphic).

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